FINAL REVIEW

Taylor series

Tn(x) $= \sum_{k=0}^{n} \frac{f^{(k)}(b)}{k!}(x-b)^k$

Error $= |f(x) - T_n(x)| \leq \frac{M}{(n+1)!}|x-b|^{n+1}$

* where $|f^{(n+1)}(x)| \leq M$
on the interval

steps to solving error on given interval $I = [c, d]$

1. Find the next derivative $(f^{(n+1)}(x))$
2. Find M by finding the biggest that $|f^{(n+1)}|$ will ever get on interval
3. plug M into error formula

Taylor Series! * know all of the listed series

$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(b)(x-b)^k = \lim_{n\to\infty} T_n(x)$

open interval of convergene

$\int$ To integrate a Taylor series $\Rightarrow$ change x^k to $\frac{1}{k+1}x^{k+1}$

$\frac{d}{dx}$ to differentiate a taylor series $\Rightarrow$ change x^k to kx^{k-1}

Substitution Questions ⇒ rearrange to fit one of the known series and fill in x

Combining Questions ⇒ add series, smallest convergence sticks

Integrating Questions

EXAM 2 REVIEW

13.4 – 14.3

13.4 – Motion in Space

t = time $r(t) = \langle x(t), y(t), z(t) \rangle$ = position

$v(t) = r'(t)$ = velocity vector

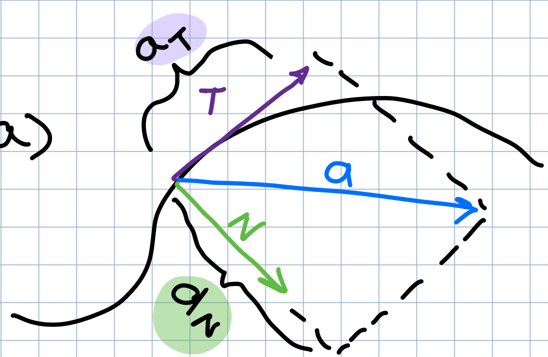
$|v(t)| = |r'(t)|$ = speed function

$a(t) = r''(t)$ = acceleration vector

components of acceleration

$$a_T = \frac{r'(t) \cdot r''(t)}{|r'(t)|} \qquad a_N = \frac{|r'(t) \times r''(t)|}{|r'(t)|}$$

$a_T = \text{Comp}_T(a)$



$a_N = \text{Comp}_N(a)$

*application

Newton's second law $\Rightarrow$ $F(t) = ma(t)$

force vector = mass (acceleration vector)

14.1 – Functions of several variables

$z = f(x,y) \Rightarrow z$ is height above xy-plane

* if one variable is fixed then only 2 variables
  ex: $z = f(4,y) \rightarrow x=4$, y changes + function of y

*Domain restriction rules:

$z = \sqrt{\text{Blah}} \Rightarrow \text{Blah} \geq 0$ $z = \sin^{-1}(\text{Blah})$

$z = \dfrac{1}{\text{Blah}} \Rightarrow \text{Blah} \neq 0$ $z = \cos^{-1}(\text{Blah})$

$z = \ln(\text{Blah}) \Rightarrow \text{Blah} > 0$ $-1 \leq \text{Blah} \leq 1$

* to graph level curves, fix z at different values

14.3 - Partial Derivatives

f_x = slope in the x direction

f_y = slope in the y direction

when facing parallel to x-axis ...
- if f_x is negative, walking down hill
- if f_x is positive, walking up hill

Same for y

f_{xx} = concavity in x-direction

f_{yy} = concavity in y-direction

f_{xy} = mixed partial = f_{yx} ⇐ always the same

14.4 - Tangent Planes

the plane that contains all tangent lines to a surface at a point

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

tangent plane approx./ linear approximation

$f(x,y) \approx L(x,y) = f(x_0,y_0) + f_x(x_0 y_0)(x-x_0) + f_y(x_0 y_0)(y-y_0)$

if (x,y) is "near" (x_0, y_0)

14.7 - Max and Min Values

be able to find critical points and classify them + find absolute max + min

* critical point ⇒ $f_x(a,b) = 0$ and $f_y(a,b) = 0$ or where either DNE

* Second Derivative test

(a,b) = critical point; $D = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2$

$D > 0$ and $f_{xx} > 0$ = local min

$D > 0$ and $f_{xx} < 0$ = local max

$D < 0$ = saddle point $D = 0$ ⇒ inconclusive

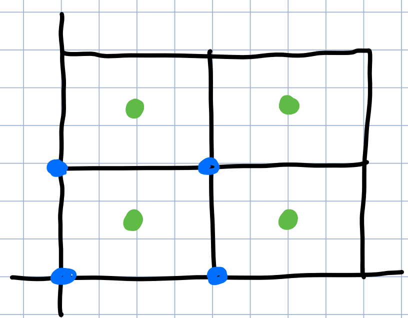
* absolute max/min occur @ critical point or on boundary of given region
  (1) Find critical values and plug into $f(x,y)$
  (2) Find formulas for each part of boundary and find values of $f(x,y)$ at these points
  (3) Absolute max = biggest output
      Absolute min = smallest output

* applied optimization

15.1 - Double Integrals over Rectangular Regions

$\iint_R f(x,y)\,dA$ = the volume under the surface and above the xy-plane

approximation



m = columns
n = rows
> compute area of one rectangle

volume = (height) (ΔA) = $f(x,y)\Delta A$

lower left endpoints

midpoint rule

Sum up all volumes

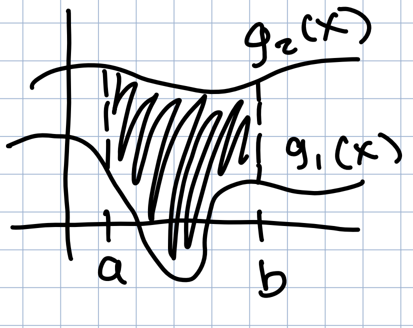
Average value = $\frac{1}{\text{area } R} \iint_R f(x,y)\,dA$

15.2 - Iterated Integrals

$\int_a^b \int_c^d f(x,y)\,dy\,dx$ = $a \le x \le b$ $c \le y \le d$
= $R = [a,b] \times [c,d]$

Rectangles ⟶

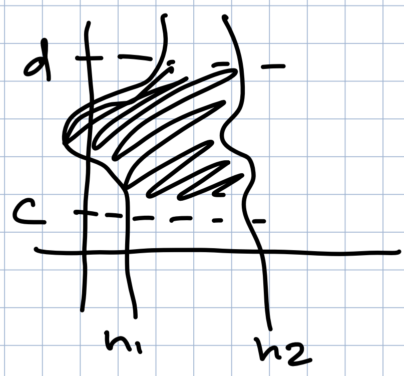
General ⟶ "Switch the order of integration"



Top/Bottom regions

$a \le x \le b \qquad g_1(x) \le y \le g_2(x)$

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$



Right/Left regions

$c \le y \le d \qquad h_1(y) \le x \le h_2(y)$

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

Steps to solve:

(1) Find integrand ⇒ solve for z, if there are two, set up two double integrals and subtract @ end

(2) Region ⇒ graph on xy-plane, z's intersect

10.3 - Polar Coordinates

$(r, \theta) \qquad x = r\cos\theta \quad y = r\sin\theta \quad x^2 + y^2 = r^2 \quad \tan\theta = \frac{y}{x}$

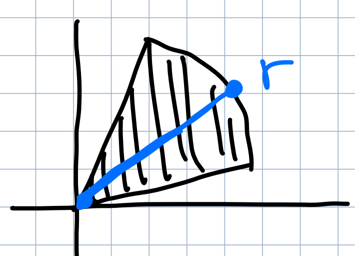
Plot points!

15.3 - Double Integrals in Polar Coordinates

$$\iint_R f(x,y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

"Inside-outside"

$$\iint_D 1\, dA = \text{"Area of D"}$$



EXAM 1 REVIEW

12.1 – 13.3

12.1 – 3D Coordinate Systems

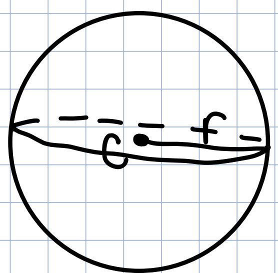
Point $P = (a, b, c)$

distance from... (yz, xz, xy)

$\mathbb{R}^2$ = 2D set of points vs. $\mathbb{R}^3$ = 3D set of points

Distance between two points =

$$\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$



$$\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} = r$$

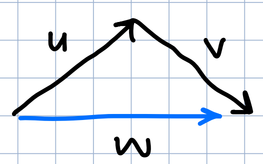
→ turns into equation of sphere

Center $= (x_0, y_0, z_0)$

Points @ distance r from $C = (x, y, z)$

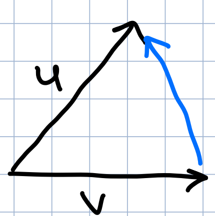
12.2 – Vectors

Adding vectors



$u + v = w$

Subtracting vectors



$u - v = w$

$\langle 1, m \rangle$ = a vector parallel to any line with slope m

• 2 non-zero vectors are parallel if they are scalar multiples

↓ if negative, opp. direction

$A(x_1, y_1, z_1)$ $B(x_2, y_2, z_2)$

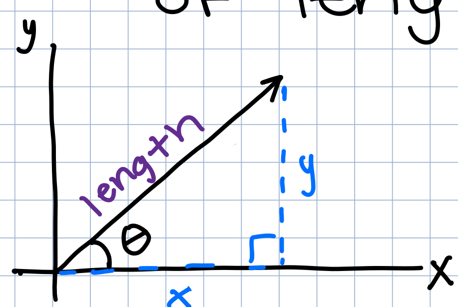
$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$

length / magnitude =

$$\sqrt{x^2 + y^2 + z^2}$$

→ use to scale unit vector of length 1

$$\frac{1}{|v|} \langle v \rangle$$



12.3 - The Dot Product

$a\langle a_1, a_2, a_3\rangle$ $\quad b\langle b_1, b_2, b_3\rangle$

$a \cdot b = a_1b_1 + a_2b_2 + a_3b_3$

result = scalar #, not vector

If θ is the angle between two non-zero vectors when drawn tail to tail...

$a \cdot b = |a||b| \cos\theta$

If a and b are perpendicular... $a \cdot b = 0$

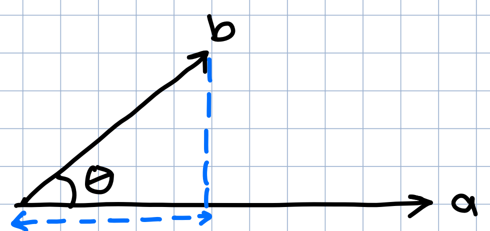
If a and b are parallel and in same direction... $a \cdot b = |a||b|$

→ opposite direction... $a \cdot b = -|a||b|$

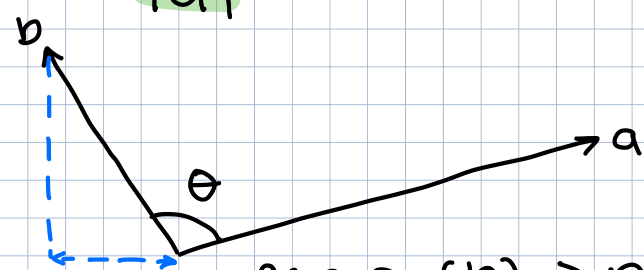
Projections

- the scalar/component projection of b onto a is the length of the projection...

$$\text{comp}_a(b) = \frac{a \cdot b}{|a|}$$



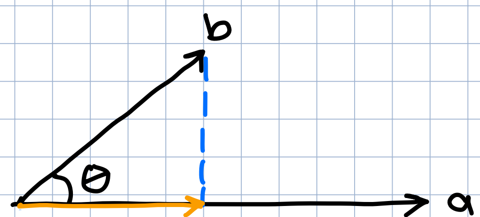
$\text{comp}_a(b) \rightarrow$ positive



$\text{comp}_a(b) \rightarrow$ negative

- the vector projection of b onto a is the actual vector you get from the projection...

$$\text{proj}_a(b) = \left(\frac{a \cdot b}{|a|^2}\right)a$$



↳ vector!

$\text{Work} = F \cdot D$

12.4 - The Cross Product

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)i - (a_1b_3 - a_3b_1)j + (a_1b_2 - a_2b_1)k$$

vector! Not scalar!

is orthogonal to a and b

use as a check!

area of parallelogram =

$|a \times b| = |a||b| \sin\theta$

OR

$|a|h = |a||b| \sin\theta$

area of triangle =

$\frac{1}{2}|a \times b|$

2 non-zero vectors are parallel if...

$a \times b = 0$

12.5 - Lines and Planes

Lines

Let (x_0, y_0, z_0) be a point on the line and let $\langle a, b, c\rangle$ be any vector parallel...

parametric eq =

$x = x_0 + at$

$y = y_0 + bt$

$z = z_0 + ct$

need
- point
- direction

symmetric eq =

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

- Two lines are parallel if their direction vectors are parallel

Planes

Let (x_0, y_0, z_0) be a point on a plane and let $\langle a, b, c\rangle$ be any vector normal...

standard form =

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

vector form =

$\langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$

- Two planes are parallel if their normal vectors are parallel
- If 2 planes are not parallel, they must intersect to form a line

- Two lines intersect if they have an (x, y, z) point in common

$\Downarrow$

use different parameters when you combine

- Two lines are skew if they don't intersect and aren't parallel

$\Downarrow$

$\vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1||\vec{n}_2| \cos\theta$

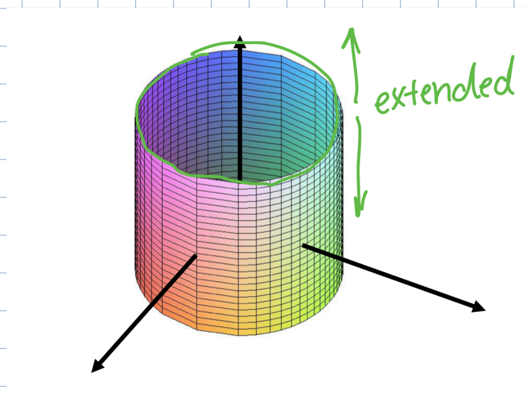
The planes are orthogonal if their normal vectors are orthogonal

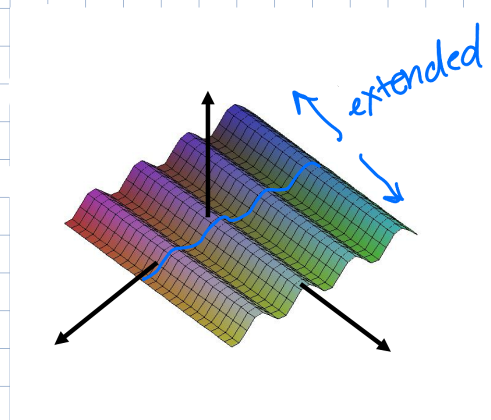
12.6 - Cylinders and Quadratic Surfaces

Cylinders – If one variable is absent

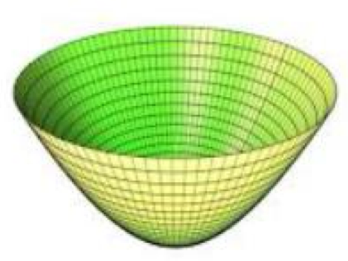
$x^2 + y^2 = 1$ circular cylinder!

$z = \cos(x)$ cosine cylinder!





Quadratic surfaces use tracing!



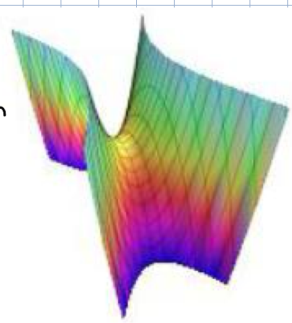
$3x^2 + 5y^2 \rightarrow$ when z is fixed, ellipse in 2D

Elliptical/Circular Paraboloid

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

(*ex*: z = 3x² + 5y²)

if hyperbola in 2D



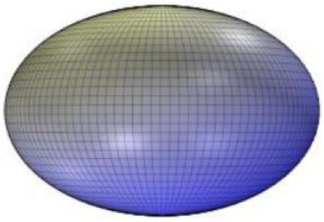
Hyperbolic Paraboloid

$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

(*ex*: y = 2x² – 5z²)

When all <u>three</u> variables are squared….

all positive

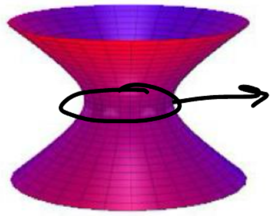


Ellipsoid/Sphere

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

(*ex*: $3x^2 + 5y^2 + z^2 = 3$)

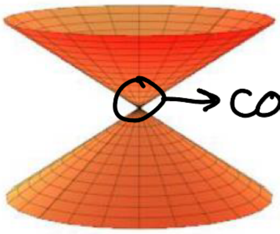
one negative →



Hyperboloid of One Sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$ Positive

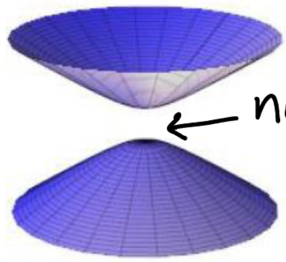
(*ex*: $x^2 - y^2 + z^2 = 10$)



Circular/Elliptical Cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$ zero

(*ex*: $z^2 = x^2 + y^2$)



Hyperboloid of Two Sheets

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$$ negative

(*ex*: $x^2 + y^2 - z^2 = -4$)

13.1 - Vector Curves

parametric form: $x=f(t)$, $y=g(t)$, $z=h(t)$

vector form: $r(t) = \langle f(t), g(t), h(t) \rangle$

↳ position vector that points from origin to curve

To visualize 3D curves...

1. Find surface/path of motion by eliminating t variable
2. Plot points

with circular motion...

$\sin^2(t) + \cos^2(t) = 1$ so if...

$x = r\cos(t)$, $y = r\sin(t)$ ⟶ $x^2 + y^2 = r^2$

Intersections

① Intersecting a curve and a surface

↳ combine conditions!

Find the intersection of

$$x = t, y = \cos(\pi t), z = \sin(\pi t)$$

and

$$(x^2 - y^2 - z^2 = 3.) \cdot -1$$

$$-x^2 + y^2 + z^2 = -3$$

Hyperboloid of 2 sheets

$$y^2 + z^2 = 1$$

$$t^2 - \cos^2(\pi t) - \sin^2(\pi t) = 3$$

$$t^2 - 1 = 3$$

$$t^2 = 4$$

$$t = \pm 2$$

intersection points

$$\boxed{\begin{array}{l}(x,y,z) = (2,1,0)\\(x,y,z) = (-2,1,0)\end{array}}$$

② Intersecting two curves

↳ use different parameters + combine!

Given:

$$\boldsymbol{r}_1(t) = \langle 2t, 3t^2, 2t^3 \rangle$$
$$\boldsymbol{r}_2(t) = \langle 2 - 2t, 3 + 3t, 2 - 6t \rangle$$

Find the (x,y,z) point(s) at which the **paths** of the two particles described cross.

① $2t = x = 2 - 2u \longrightarrow u = 1 - t$

② $3t^2 = y = 3 + 3u \rightarrow t^2 = 1 + u \rightarrow t^2 = 1 + 1 - t$

③ $2t^3 = z = 2 - 6u$

$$t^2 = 2 - t$$
$$t^2 + t - 2 = 0$$
$$(t-1)(t+2) = 0$$
$$t = 1, -2$$

$u = 1 - 1$ $\quad u = 1 + 2$

$$u = 0, 3$$

@ $t = 1, u = 0$

$x = -4$

$y = 12$

$z = 2$

@ $t = -2, u = 3$

$x = 2$

$y = 3$

$z = -16$

$$\boxed{(-4, 12, 2) \text{ and } (2, 3, -16)}$$

③ Intersecting two surfaces
↳ parameterize the curve by:

a) Let one variable be t, solve for others in terms of t

OR if circle/ellipse

b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \longleftrightarrow \begin{array}{l} x = a\cos(t) \\ y = b\sin(t) \end{array}$

13.2 - Calculus for vector curves

If $r(t) = \langle x(t), y(t), z(t) \rangle$ then...

$r'(t) = \langle x'(t), y'(t), z'(t) \rangle$ = a vector tangent to the curve at t

$T(t) = \dfrac{1}{|r'(t)|} r'(t)$ = unit tangent vector

$$\int r(t)\,dt = \left\langle \int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt \right\rangle$$

(each has its own C)

Equation for the tangent line to $(x(t), y(t), z(t))$ at $t=a$...

$$\begin{aligned} x &= x(a) + x'(a)u \\ y &= y(a) + y'(a)u \\ z &= z(a) + z'(a)u \end{aligned}$$

$r'(a)$ = direction vector

* If you need the angle of intersection of two curves, you need the angle between their tangent vectors at their intersection

13.3 - Tools for Analyzing Vector Curves

Arc Length

If $r(t) = \langle x(t), y(t), z(t) \rangle$ then...

distance traveled from $t=a$ to $t=b$ =

$$\int_a^b |r'(t)|\,dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

Reparameterize in terms of Arc Length:

1. compute + simplify arc length function
2. solve for t in terms of s
3. Rewrite curve in terms of s

unit Tangent + Normal vectors

speed = $\frac{\text{distance}}{\text{time}}$

$T(t) = \frac{1}{|r'(t)|} r'(t)$ → unit tangent

$N(t) = \frac{1}{|T'(t)|} T'(t)$ → principal unit normal (inward pointing)

Curvature

↳ measures how quickly the direction of the unit tangent vector is changing with respect to distance

$$K(t) = \left|\frac{dT}{ds}\right| = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}}$$

(3D) (2D)

* the radius of curvature = radius of circle that best fits @ point = reciprical of curvature